Arithmetic circuits:
a chasm at depth three

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Based on joint work with

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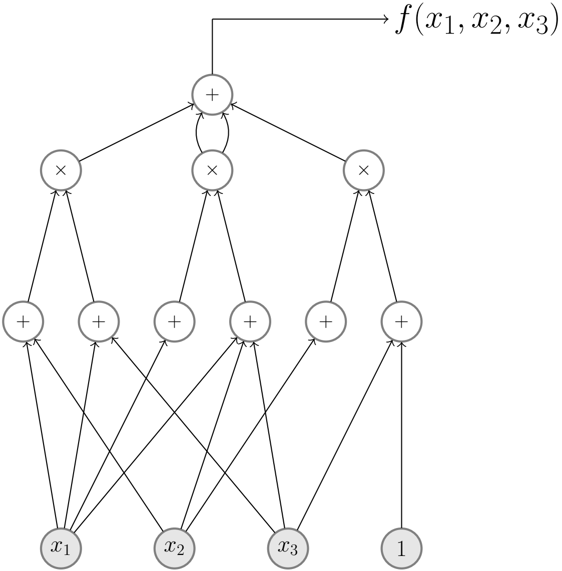
FOCS - 2013
UC Berkeley

Arithmetic Circuits



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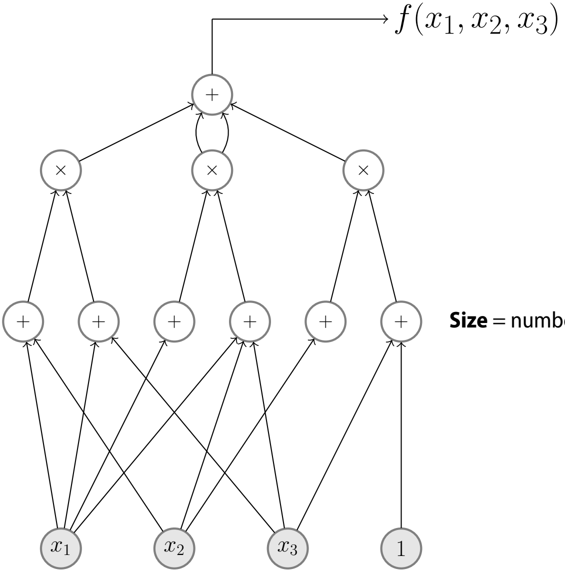
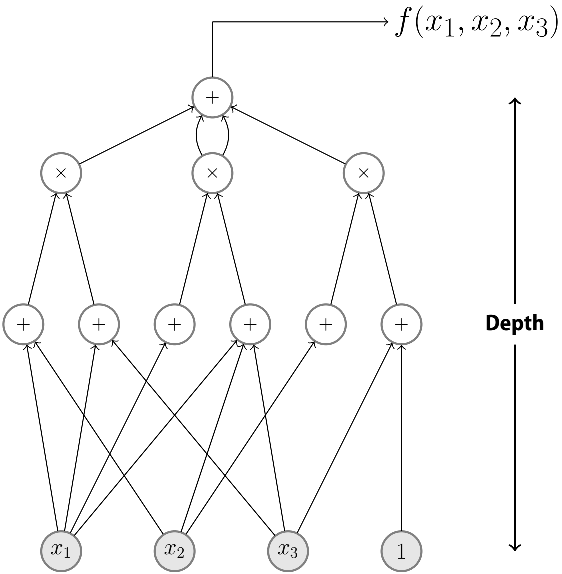


Size = number of gates

Arithmetic Circuits



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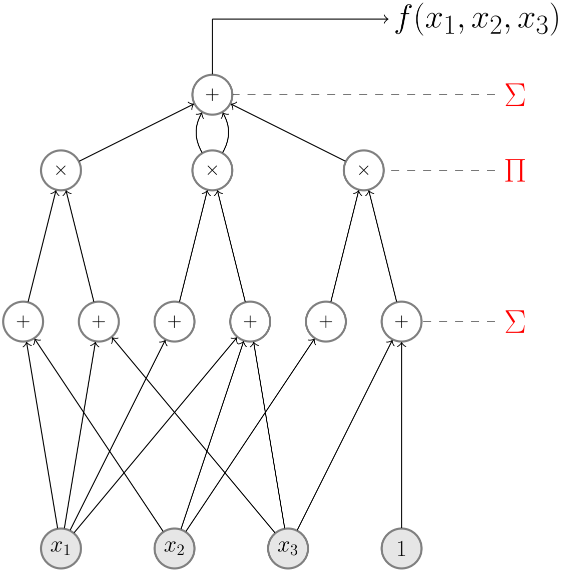

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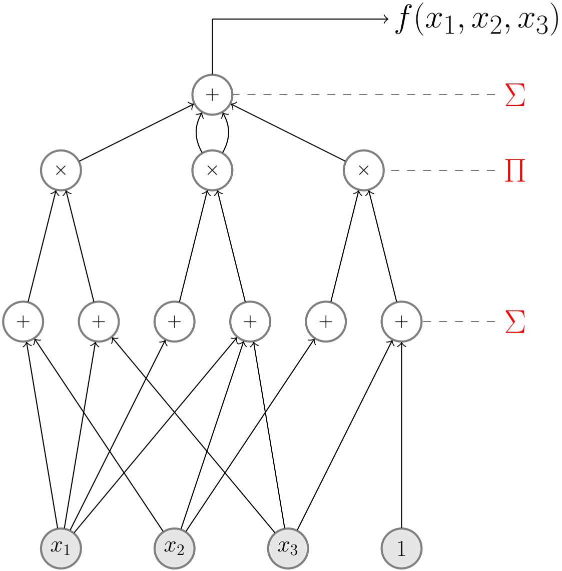

Root is always $(+)$

An algebraic analogue of "P vs NP"

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$$\mathsf{VP} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} P(x_1, \ldots, x_n) \ : \text{poly}(n) \text{ degree} \\ \text{computable by poly}(n)\text{-sized circuits} \end{array} \right\}$$

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Det_n vs Perm_n ⟷ [Valiant-79] ⟷ VP vs VNP

An algebraic analogue of "P vs NP"

Det_n

$$\text{Det}\left(\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}\right)$$

Perm_n

$$\text{Perm}\left(\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}\right)$$

Det_n		VP
	[Valiant-79]	
vs	⟷	vs
Perm_n		VNP

An algebraic analogue of "P vs NP"

"The determinant of this conjecture will be permanently famous..."

- Neeraj Kayal

Det_n vs Perm_n ←[Valiant-79]→ VP vs VNP

Known lower bounds (before 2012)

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Depth-3 circuits	$\widetilde{\Omega}(n^4)$	[Shpilka-Wigderson-01]
Homogeneous Depth-3 circuits	$2^{\Omega(n)}$	[Nisan-Wigderson-97]
Depth-3 circuits over finite fields	$2^{\Omega(n)}$	[Grigoriev-Karpinski-98]

Known lower bounds (before 2012)

Model	**Lower bound**	
Multilinear formula	$n^{\Omega(\log n)}$	[Raz-09]
Constant depth, *multilinear* formula	$2^{\tilde{\Omega}(n^{1/d})}$	[Raz-Yehudayoff-09]
Monotone Circuits	$2^{\Omega(n)}$	[Jerrum-Snir-82]

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Summary: No lower bounds known beyond depth-3, unless other restrictions are imposed.

Depth reduction

Reduction to $\log^2$-depth

Theorem ([Valiant-Skyum-Berkowitz-Rackoff-83])

Arithmetic circuits	$\longrightarrow$	*Log^2-depth circuits*
of "small" size		*of "small" size*

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$$\text{VP} = \text{VNC}^2$$

Reduction to depth-4

Theorem ([Agrawal-Vinay-08, Koiran-12, Tavenas-13])

Arithmetic circuits		*Depth-4 circuits*
	$\longrightarrow$	
of "small" size		*of "not-too-large" size*

Reduction to depth-4

Theorem ([Agrawal-Vinay-08, Koiran-12, Tavenas-13])

Arithmetic circuits	$\longrightarrow$	*Depth-4 circuits*
of poly(n, d) *size*		*of* $n^{O(\sqrt{d})}$ *size*

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And last year ...

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Lower bound on size of depth-4 circuits*

[Gupta-K.-Kayal-Saptharishi] Perm$_d$ (or Det$_d$) $2^{\Omega(\sqrt{d})}$

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[Gupta-K.-Kayal-Saptharishi]	Perm_d (or Det_d)	$2^{\Omega(\sqrt{d})}$
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Escalator at depth-4

Theorem ([Agrawal-Vinay-08, Koiran-12, Tavenas-13])

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Squashing it further

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Is a similar depth-reduction possible to depth-3?

But...

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No depth-3 circuit for determinant of size $2^{O(n)}$ was known.
The permanent however has *Ryser's formula* of size $2^{O(n)}$.

$$\mathrm{Perm}_n = \sum_{S \in [n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j \in S} x_{ij}$$

Theorem ([Gupta-K.-Kayal-Saptharishi])

Arithmetic circuits	over $\mathbb{Q}$ $\longrightarrow$	Depth-3 circuits
poly(n, d) size		of $n^{O(\sqrt{d})}$ size

... it's true!

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Corollary

A depth-3 circuit for Det$_d$ *of size* $d^{O(\sqrt{d})}$ *over* $\mathbb{Q}$.

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Theorem ([Gupta-K.-Kayal-Saptharishi])

Arithmetic circuits poly(n, d) *size* $\xrightarrow{\text{over } \mathbb{Q}}$ *Depth-3 circuits of* $n^{O(\sqrt{d})}$ *size*

Corollary

A depth-3 circuit for Det_d *of size* $d^{O(\sqrt{d})}$ *over* $\mathbb{Q}$.

Note: Resulting depth-3 circuit is *heavily non-homogeneous*, with degrees going up to $n^{O(\sqrt{d})}$.

Reduction to Depth-3 Circuits

Road map

general circuit
of size $n^{O(1)}$

general circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuits

Road map

general circuit of size $n^{O(1)}$ → [AV08], [Koi12], [Tav13] → $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuits

↓

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ circuits

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$\sum\prod\sum$ circuits

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$\downarrow$ [Fischer]'s lemma

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↓ [Fischer]'s lemma (Only over $\mathbb{Q}$, $\mathbb{R}$ etc.)

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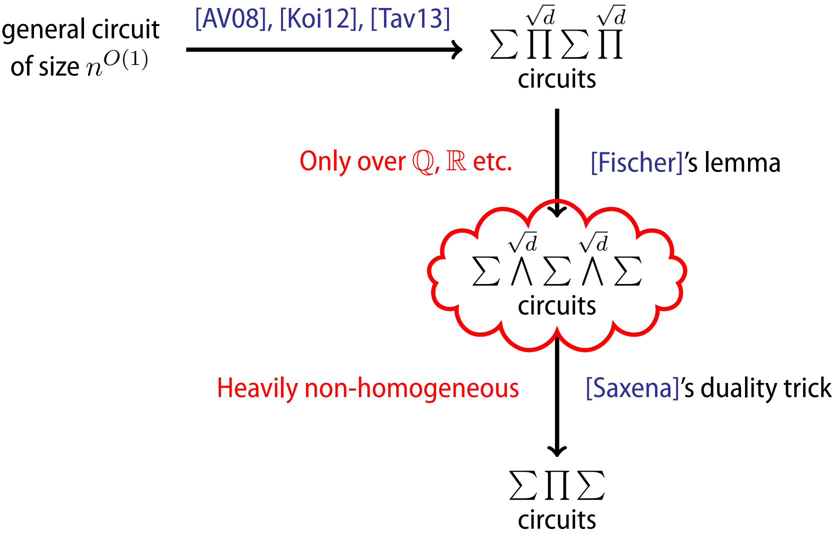
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Where did $\Sigma \wedge \Sigma \wedge \Sigma$ come from?

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$$C = \sum_{i=1}^{s} Q_{i1} \cdot Q_{i2} \cdots Q_{i\sqrt{d}}$$

where, $\deg(Q_{ij}) \leq \sqrt{d}$

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A possibly simpler circuit,

$$C_{\text{simpler}} \quad = \quad \sum_{i=1}^{s} Q_i^{\sqrt{d}}$$

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Turns out: $\sum\bigwedge^{\sqrt{d}}\sum\prod^{\sqrt{d}}$ is as powerful as $\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}}$

Step 1: $\Sigma\Pi\Sigma\Pi \longrightarrow \Sigma\wedge\Sigma\Pi$

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$$4\,x_1x_2 = (x_1+x_2)^2 - (x_1-x_2)^2$$

Step 1: $\Sigma\Pi\Sigma\Pi \longrightarrow \Sigma\wedge\Sigma\Pi$

$$
\begin{aligned}
24\, x_1x_2x_3 \quad = \quad & (x_1 + x_2 + x_3)^3 \\
& - (x_1 - x_2 + x_3)^3 \\
& - (x_1 + x_2 - x_3)^3 \\
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[Fischer]:

$$d! \cdot 2^{d-1} \cdot (x_1 x_2 \cdots x_d) = \sum_{S \subseteq [d] \setminus \{1\}} (-1)^{|S|} \left(\sum_{j \notin S} x_j - \sum_{j \in S} x_j \right)^d$$

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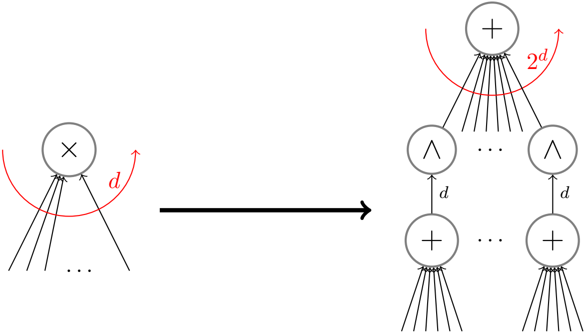

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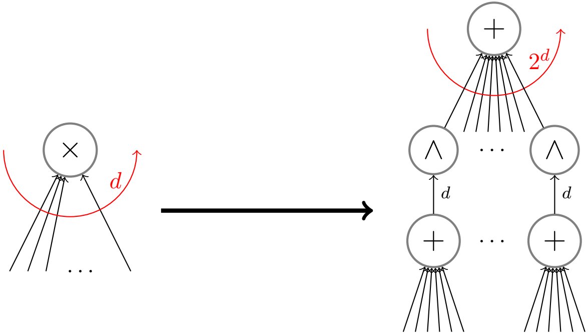

$$\prod^{d} \quad \longrightarrow \quad \sum^{2^d} \bigwedge^{d} \sum^{d}$$

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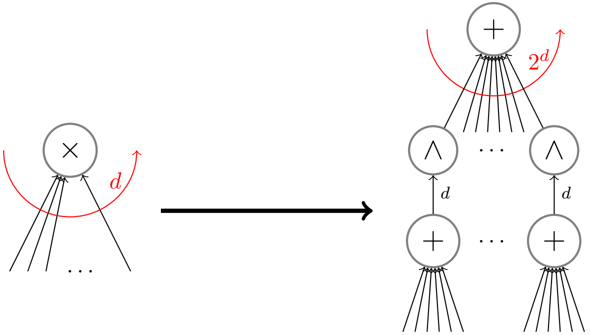

$$\prod^{d} \rightarrow \sum^{2^d} \bigwedge^{d} \sum^{d}$$

$$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$$ of size s

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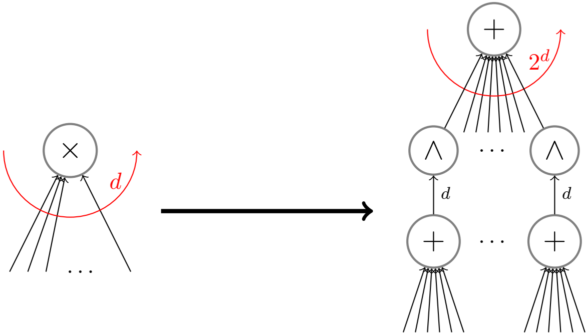

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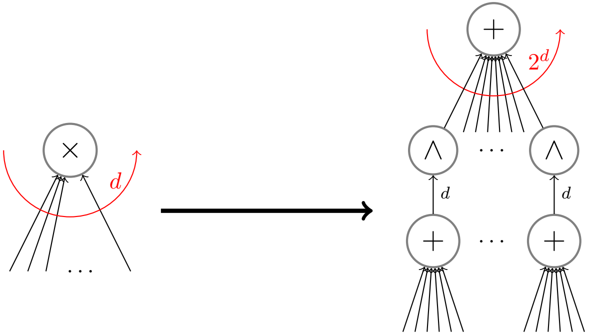

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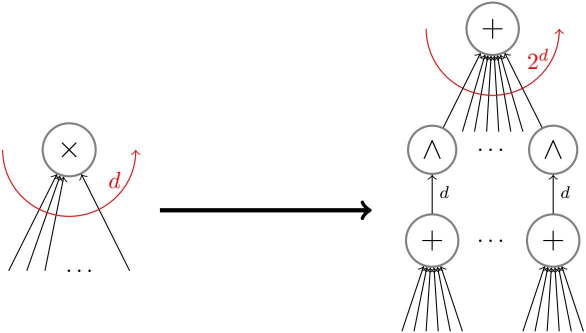

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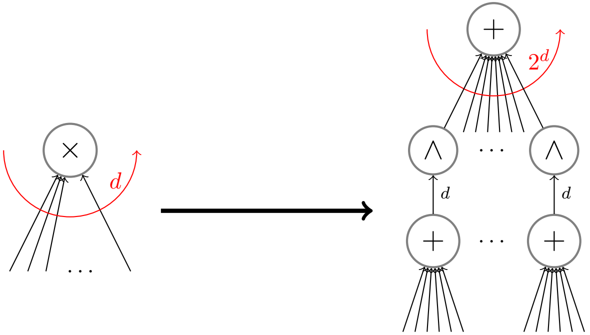

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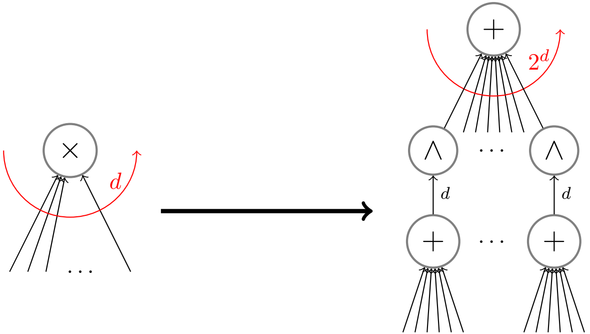

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$$\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}} \text{ of size } s \longrightarrow \sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum \text{ of size } 2^{O(\sqrt{d})}\cdot s$$

Road map

general circuit of size $n^{O(1)}$ → [AV08], [Koi12], [Tav13] → $\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}}$ of size $n^{O(\sqrt{d})}$

↓

$\sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum$ of size $n^{O(\sqrt{d})}$

Road map

general circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}}$ of size $n^{O(\sqrt{d})}$

$\downarrow$ ✓

$\sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum$ of size $n^{O(\sqrt{d})}$

Road map

general circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ of size $n^{O(\sqrt{d})}$

$\downarrow$ ✓

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ of size $n^{O(\sqrt{d})}$

$\exists$ efficient reduction : $\sum \prod \sum \quad \longrightarrow \quad \sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$

Road map

general circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}}$ of size $n^{O(\sqrt{d})}$

$\downarrow$ ✓

$\sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum$ of size $n^{O(\sqrt{d})}$

$\exists$ efficient reduction : $\sum\prod\sum \longrightarrow \sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum$

$$n^{O(\sqrt{d})} \longrightarrow n^{O(\sqrt{d})}$$

Road map

general circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ of size $n^{O(\sqrt{d})}$

$\downarrow$ ✓

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ of size $n^{O(\sqrt{d})}$

$\exists$ efficient reduction : $\sum \prod \sum \longrightarrow \sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$

$n^{O(\sqrt{d})} \longrightarrow n^{O(\sqrt{d})}$

Can you : $\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum \xrightarrow{?!} \sum \prod \sum$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$$

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ℓ

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma \longrightarrow \Sigma\Pi\Sigma$

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$$

$$\ell^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$$

$$\ell_1^{\sqrt{d}} + \ldots + \ell_s^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$$

$$\left(\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$\sum \overset{\sqrt{d}}{\bigwedge} \sum \overset{\sqrt{d}}{\bigwedge} \sum$$

$$\sum_i \left(\ell_{i1}^{\sqrt{d}} + \ldots + \ell_{is}^{\sqrt{d}}\right)^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$C \quad = \quad \sum_i \left(\ell_{i1}^{\sqrt{d}} + \cdots + \ell_{is}^{\sqrt{d}}\right)^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

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Lemma ([Saxena])

There exists univariate polynomials f_{ij}'s of degree at most d such that

$$(x_1 + \cdots + x_s)^d \quad = \quad \sum_{i=1}^{sd+1} f_{i1}(x_1) \cdot f_{i2}(x_2) \cdots f_{is}(x_s)$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$(x_1 + \cdots + x_s)^{\sqrt{d}} = \sum_i^{\mathsf{poly}(s,d)} f_{i1}(x_1) \cdot f_{i2}(x_2) \cdots f_{is}(x_s)$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} \quad = \quad \sum_{i}^{\mathsf{poly}(s,d)} f_{i1}\left(\ell_1^{\sqrt{d}}\right) \cdot f_{i2}\left(\ell_2^{\sqrt{d}}\right) \cdots f_{is}\left(\ell_s^{\sqrt{d}}\right)$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} = \sum_i^{\mathsf{poly}(s,d)} f_{i1}\left(\ell_1^{\sqrt{d}}\right) \cdot f_{i2}\left(\ell_2^{\sqrt{d}}\right) \cdots f_{is}\left(\ell_s^{\sqrt{d}}\right)$$

$$= \sum_i^{\mathsf{poly}(s,d)} \tilde{f}_{i1}(\ell_1) \cdot \tilde{f}_{i2}(\ell_2) \cdots \tilde{f}_{is}(\ell_s)$$

where $\tilde{f}_{ij}(t) := f_{ij}(t^{\sqrt{d}})$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\begin{aligned}\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} &= \sum_i^{\mathsf{poly}(s,d)} f_{i1}\left(\ell_1^{\sqrt{d}}\right) \cdot f_{i2}\left(\ell_2^{\sqrt{d}}\right) \cdots f_{is}\left(\ell_s^{\sqrt{d}}\right) \\ &= \sum_i^{\mathsf{poly}(s,d)} \tilde{f}_{i1}(\ell_1) \cdot \tilde{f}_{i2}(\ell_2) \cdots \tilde{f}_{is}(\ell_s)\end{aligned}$$

Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

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$$= \sum_i^{\mathsf{poly}(s,d)} \tilde{f}_{i1}(\ell_1) \cdot \tilde{f}_{i2}(\ell_2) \cdots \tilde{f}_{is}(\ell_s)$$

Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial that can be factorized over $\mathbb{C}$:

$$\tilde{f}_{ij}(t) = \prod_{k=1}^{d} (t - \zeta_{ijk})$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

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... a $\Sigma\Pi\Sigma$ circuit of poly(s, d) size.

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma \longrightarrow \Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\begin{aligned}\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} &= \sum_i^{\mathsf{poly}(s,d)} f_{i1}\left(\ell_1^{\sqrt{d}}\right) \cdot f_{i2}\left(\ell_2^{\sqrt{d}}\right) \cdots f_{is}\left(\ell_s^{\sqrt{d}}\right) \\ &= \sum_i^{\mathsf{poly}(s,d)} \tilde{f}_{i1}(\ell_1) \cdot \tilde{f}_{i2}(\ell_2) \cdots \tilde{f}_{is}(\ell_s) \\ &= \sum_i^{\mathsf{poly}(s,d)} \prod_{j=1}^{s} \prod_{k=1}^{d} (\ell_j - \zeta_{ijk})\end{aligned}$$

... a $\Sigma\Pi\Sigma$ circuit of poly(s, d) size and degree sd.

Putting it together

general circuit
of size $n^{O(1)}$

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general circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}}$ circuit of size $n^{O(\sqrt{d})}$

Putting it together

general circuit of size $n^{O(1)}$

[AV08], [Koi12], [Tav13] →

$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuit of size $n^{O(\sqrt{d})}$

↓

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ circuits of size $n^{O(\sqrt{d})} \cdot 2^{O(\sqrt{d})}$

Putting it together

general circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuit of size $n^{O(\sqrt{d})}$

$\downarrow$

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ circuits of size $n^{O(\sqrt{d})}$

Putting it together

general circuit of size $n^{O(1)}$ —[AV08], [Koi12], [Tav13]→ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuit of size $n^{O(\sqrt{d})}$

↓

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ circuits of size $n^{O(\sqrt{d})}$

←

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Putting it together

general circuit of size $n^{O(1)}$ → [AV08], [Koi12], [Tav13] → $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuit of size $n^{O(\sqrt{d})}$

↓

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ circuits of size $n^{O(\sqrt{d})}$ → $\sum \prod \sum$ circuits of size $n^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuit of size $n^{O(\sqrt{d})}$

$\downarrow$

$\sum \prod \sum$ circuits of size $n^{O(\sqrt{d})}$ $\longleftarrow$ $\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ circuits of size $n^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ hom. circuit of size $n^{O(\sqrt{d})}$

$\downarrow$

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ circuits of size $n^{O(\sqrt{d})}$ $\longrightarrow$ $\sum \prod \sum$ circuits of size $n^{O(\sqrt{d})}$

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general hom. circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ hom. circuit of size $n^{O(\sqrt{d})}$

$\downarrow$

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ hom. circuits of size $n^{O(\sqrt{d})}$ $\longrightarrow$ $\sum \prod \sum$ circuits of size $n^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size $n^{O(1)}$ $\xrightarrow{\text{[AV08], [Koi12], [Tav13]}}$ $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ hom. circuit of size $n^{O(\sqrt{d})}$

$\downarrow$

$\sum \prod \sum$ **non-hom.** circuits of size $n^{O(\sqrt{d})}$ $\longleftarrow$ $\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ hom. circuits of size $n^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size $n^{O(1)}$

[AV08], [Koi12], [Tav13] $\longrightarrow$

$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ hom. circuit of size $n^{O(\sqrt{d})}$

$\downarrow$

$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$ hom. circuits of size $n^{O(\sqrt{d})}$

$\longleftarrow$

$\sum \prod \sum$ **non-hom.** circuits of size $n^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Suffices to show this:

Find an explicit $f(x_1, \ldots, x_n)$ such that if

$$f(x_1, \ldots, x_n) = Q_1^{\sqrt{d}} + \ldots + Q_s^{\sqrt{d}}$$
$$\text{where} \quad \deg Q_i \leq \sqrt{d} \quad \text{for all } i$$

then $s = n^{\omega(\sqrt{d})}$.

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How hard can this be !?

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Thank you!