Group Theory in Rubik's Cube

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Overview

- Notations
- Counting *Rubik's cube* configurations...
- Quick review of *Group Theory*.
- The Rubik's Cube Group.
- More Group Theory and representation of Rubik's Cube Configurations
- Counting valid Rubik's Cube Configurations.
- Optimal solution of the Rubik's Cube

Notations

- The Rubik’s cube is composed of 27 small cubes, which are typically called *cubies*. 26 of these cubies are visible (infact, the 27th cubie doesn’t actually exist).
- Types of cubies :
    - 8 corner cubies
    - 12 edge cubies
    - 6 center cubies
- *Cubicles* describe the space in which the Cubies lie.

Number of possible *Cube Configurations*

- $8!$ = number of ways of arranging the corner cubies
- 3^8 = number of ways of orienting the corner cubies
- $12!$ = number of ways of arranging the edge cubies
- 2^{12} = number of ways of orienting the edge cubies
- Total number of configurations = $3^8.2^{12}.8!.12!$

  = 519,024,039,293,878,272,000 = $\sim 519 \times 10^{18}$
- These configurations that can be obtained by breaking the cube and fixing it again.
- The center cubies remain where they are, even after breaking open the cube.

Number of possible *Cube Configurations*

- These configurations are theoretically possible, but that doesn’t mean that these configurations could really occur. A configuration of the Rubik’s cube is said to be *valid* if it can be achieved by a series of moves from the starting configuration. It turns out that some of the theoretically possible configurations, we have just counted, are actually not valid. Therefore, we have two goals:
  - Demonstrate that some configurations are not valid.
  - Find a set of moves that can take us from any valid configuration back to the start configuration.

Group Theory – Quick Review

- A group $(G, *)$ consists of a set G and an operation $*$ such that:
  - if $a, b \in G$, then $a*b \in G$ – Closure
  - if $a,b,c \in G$, then $a*(b*c) = (a*b)*c$ –Associativity
  - There exists $e \in G$ s.t. $g*e = e*g = g$ – Existence of Identity
  - For all $g \in G$, there exists $h \in G$, s.t. $g*h = h*g = e$ – Existence of Inverse

The Rubik's Cube Group

- The set of all possible moves on a Rubik's cube form the *Rubik's Cube Group*.
- The operation * is the simple composition rule.
- Two *moves* are considered same, if the final configuration after the moves are the same.
- Thus, this set is, in fact, similar to the set of all *valid* configurations, as discussed earlier.

Generators of a group

- Let $(G,*)$ be a group and S be a subset of G. We say that S generates G or that S is a set of generators of G if every element of G can be written as a finite product (under the group operation) of elements of S and their inverses

$$G = <S>$$

- In case of a finite group G, S generates G iff every element is expressible as finite product of elements of S (that is, inverses of S are not required)

Generators of the *Rubik's Cube Group*

- $S = \{U, D, L, R, F, B\} \subset G$
  - U : clockwise rotation of top face
  - D : clockwise rotation of bottom face
  - L : clockwise rotation of left face
  - R : clockwise rotation of right face
  - F : clockwise rotation of front face
  - B : clockwise rotation of back face
- *S is the generator of the group G*

The Symmetric Group, S_n

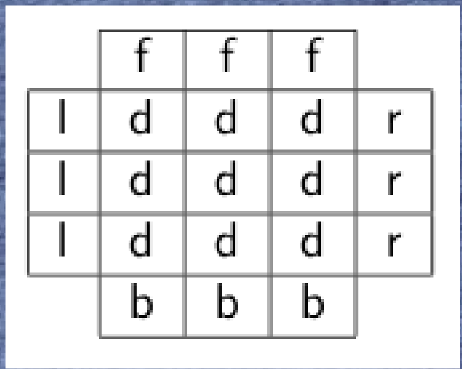
- The symmetric group on *n* letters is the set of bijections from {1, 2, . . . , n} to {1, 2, . . . , n}, with the operation of composition.
- This group is represented by as S_n
- Thus, given any configuration of the Rubik's Cube,
  - the location of the corner cubies can be represented by an element in S_8
  - the location of the edge cubies can be represented by an element in S_{12}

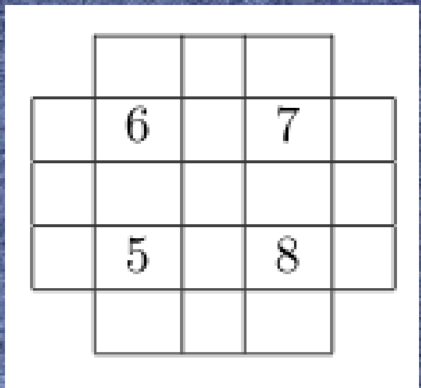
Representing
Rubik's Cube Configurations

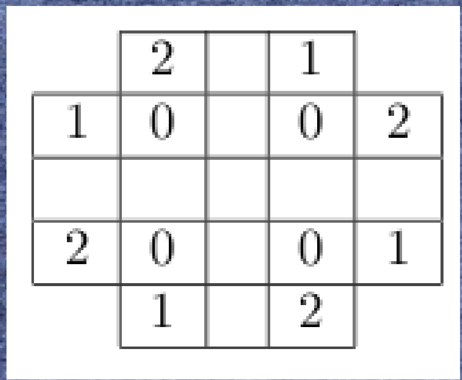
- A configuration of the Rubik’s cube is completely determined by four pieces of data:
  - the positions of the corner cubies
  - the positions of the edge cubies
  - the orientations of the corner cubies
  - the orientations of the edge cubies
- Positions of corner cubies and edge cubies, are represented by elements of S_8 and S_{12} respectively
- Claim : All elements of S_8 and S_{12} are reachable.

Representing orientations of *Cubies*

- Each corner cubie has 3 possible orientations, and we will number these orientations 0, 1, and 2.
- Figures below indicate a possible assignment of these numbers to the cube orientations.







Representing orientations of *Cubies*

- If the Rubik's cube is in any configuration, we will describe the orientations of the corner cubies as : For any i between 1 and 8, find the cubicle face labeled i. Let x_i be the number of the *cubie* face living in this *cubicle* face.
  Let x be the ordered 8-tuple (x_1 , x_2, . . . , x_7, x_8)
  - x_i is the number of clockwise twists the *cubie* 'i' is away from having its 0 face in the numbered face of the *cubicle*.
- Similarly the orientation of the 12 edge cubies can be represented by a 12-tuple (y_1, y_2, … , y_{11}, y_{12})
  - where each y_i, is either 0 or 1.

Representing *Rubik's Cube Configurations*

- Thus, any Rubik's Cube Configuration can be represented by a 4-tuple (σ, τ, x, y) :
  - $\sigma \in S_8$ – Permutations of corner cubies
  - $\tau \in S_{12}$ – Permutations of edge cubies
  - $x = (x_1, x_2, \ldots, x_7, x_8)$ – orientation of corner cubies
  - $y = (y_1, y_2, \ldots, y_{11}, y_{12})$ – orientation of edge cubies

Valid Cube Configurations

- **Theorem :** A configuration (σ, τ, x, y) is *valid* iff $\operatorname{sgn} \sigma = \operatorname{sgn} \tau$, $\Sigma\, x_i \equiv 0 \pmod 3$, and $\Sigma\, y_i \equiv 0 \pmod 2$
- The proof is not very difficult, but has a large amount of detail, which we'll skip here.
- By this theorem, the total number of valid configurations is reduced by a factor of 2x3x2 = 12.
- Thus the total number of valid configurations

  = 43,252,003,274,489,856,000 = ~43 x 10^{18}

Optimal Solution of the Cube and the Cayley Graph

- A Cayley graph, or the Cayley colour graph, is a graph that encodes the structure of a discrete group.
- Suppose, G is a group, with a generator S,
  Two vertices a and b are adjacent in the Cayley graph, iff there exists $s \in S$, such that
  either $(a = b*s)$ or $(b = a*s)$.
- **Claim :** Diameter of the Cayley graph generated by the Rubik's cube is 22.

  which means, *any cube configuration* can be solved in atmost 22 steps !!

Thank you